Journal of Geometry and Physics 9 (1992) 75–100
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

Extensions of the Poisson bracket to differential forms and multi-vector fields

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Received 10 June 1991
(Revised 23 September 1991)

The Poisson bracket defined originally on the smooth function algebra of a Poisson manifold is extended to the space of all co-exact forms of this manifold. For the extended bracket analogues of the basic constructions and formulae of the standard hamiltonian formalism are given. The Poisson bracket is extended also, in a dual way, to the space of all co-exact multi-vector fields. Finally, we define the graded Lie algebra homomorphisms connecting these extended brackets and their "differentials" as well. The method used is based on the "unification" techniques introduced by the second author.

Keywords: *Poisson bracket, Schouten–Nijenhuis bracket, Frölicher–Nijenhuis bracket, hamiltonian formalism, graded Lie algebras*
1991 MSC: 17B70, 58F05

Introduction

Let M be a Poisson manifold. Treating the exterior algebra $\Lambda^*(M)$ of differential forms on M as a natural graded (or "super") extension of the algebra $C^\infty(M)$, one could ask for the corresponding extension of the Poisson bracket defined on $C^\infty(M)$ to the whole algebra $\Lambda^*(M)$. Besides this rather general reason there exist a number of more concrete ones which lead to the same question. For instance, one of them comes from the singularity theory for solutions of partial differential equations. Namely, it is well known that the standard hamiltonian formalism describes the propagation of the so-called "wave fronts" (see refs. [1,2]) as well as as the propagation of the simplest geometric singularities of solutions of partial differential equations (see refs. [3–5]). So, it is natural to expect that the propagation of higher-order geometric singularities which were found in ref. [6] can also be described in terms of a suitably extended hamiltonian formalism.

In fact, a natural extension of the Poisson bracket to the algebra $\Lambda^*(M)$, properly, is hardly possible and we answer the above question by extending this bracket to the space $\mathscr{P}(M) = \Lambda^*(M)/d\Lambda^*(M)$ of all co-exact forms on M. For symplectic manifolds our extension coincides with that of Michor proposed in ref. [7] and, therefore, can be considered as the generalization of Michor's result to arbitrary Poisson manifolds. Our approach, however, is quite different and is based on the unified Schouten–Nijenhuis and Frölicher–Nijenhuis brackets [11]. This gives some advantages, say, in reproducing on $\mathscr{P}(M)$ all basic elements of the hamiltonian formalism and also enables us to relate naturally the extended bracket with some other ones. In particular, for a given Poisson structure Θ on M, we imbed $\mathscr{P}(M)$ into the following commutative diagram of graded Lie algebras and their homomorphisms:

$$\begin{tikzcd}
\llbracket\cdot,\cdot\rrbracket_\Theta \quad \Lambda^*(M) \arrow[r, "\Gamma_\Theta"] & \mathscr{D}_*(M) \quad \llbracket\cdot,\cdot\rrbracket \\
\{\cdot,\cdot\} \quad \Lambda^*(M)/d\Lambda^*(M) \arrow[u, "d"] \arrow[r, "\gamma_\Theta"'] & \mathscr{D}_*(M)/d_\Theta\mathscr{D}_*(M) \quad \{\cdot,\cdot\}_\Theta \arrow[u, "d_\Theta"']
\end{tikzcd}$$

The corresponding brackets are indicated at the corners. Here $\mathscr{D}_*(M)$ is the exterior algebra of multi-vector fields on M supplied with the Schouten–Nijenhuis bracket $\llbracket\cdot,\cdot\rrbracket$ and the hamiltonian differential d_Θ. The bracket $\llbracket\cdot,\cdot\rrbracket_\Theta$, which we treat according to Krasil'shchik [8] as the "dual" Schouten–Nijenhuis bracket, was introduced by Karasev [9] and independently by Kosmann-Schwarzbach and Magri [10]. The "dual" Poisson bracket $\{\cdot,\cdot\}_\Theta$ (see section 5) on co-exact (with respect to Θ) multi-vector fields seems to be introduced here for the first time.

All basic constructions of this paper are, in fact, of a rather general nature and can be applied more or less automatically to a number of situations of interest. For instance, it is straightforward to extend the Poisson bracket to the space $\mathscr{N}(M)$ of co-exact super-differential operators acting on $\Lambda^*(M)$ (see the preliminary section). However, we do not touch here upon these possibilities nor on cohomological aspects of the developed formalism, hoping to discuss it in a separate work.

Notation

Throughout this paper M denotes a smooth n-dimensional manifold. We adopt the notations $\mathscr{D}_*(M) = \sum_i \mathscr{D}_i(M)$ and $\Lambda^*(M) = \sum_i \Lambda^i(M)$ for the exterior algebras of multi-vector fields and of differential forms on M. We also set

$$B(M) = \sum_i B_i(M), \qquad B_i(M) = d\Lambda^{i-1}(M) \quad \text{(exact forms)},$$

$$\mathscr{P}(M)=\sum_i \mathscr{P}_i(M), \qquad \mathscr{P}_i(M)=\Lambda^i(M)/B_i(M) \quad (\text{"co-exact" forms}) .$$

Evidently, $\mathscr{P}(M)=\Lambda(M)/B(M)$ and $\mathscr{P}_0(M)=C^\infty(M)$. We put $\mathscr{D}(M)=\mathscr{D}_1(M)$.

All tensor products in this paper are taken over $C^\infty(M)$.

The $\Lambda^*(M)$ module $N(M)=\sum_i \Lambda^i(M)\otimes\mathscr{D}(M)$ of all vector-valued differential forms on M is supposed to be endowed with the Frölicher–Nijenhuis bracket operation, with respect to which it is a graded Lie algebra.

The Lie derivative of a differential form $\omega\in\Lambda^*(M)$ along a vector field $X\in\mathscr{D}(M)$ is denoted by $L_X(\omega)$, or, briefly, $X(\omega)$.

We use both $X\rfloor\omega$ and $i_X\omega$ for the insertion of X into ω.

If $\{K=\sum_j K^j, d\}$, $d: K^j\to K^{j+1}$, is a cochain complex, then $\operatorname{Hgr} K=\sum_i \operatorname{Hgr}^i K$ denotes the group of all graded maps of K into itself.

In this paper we work with graded objects of various types. In order to simplify the notations related to the corresponding signs we adopt the following rule: the symbol of a graded object used as the exponent of (-1) denotes the degree of that object, mod 2. For instance, the expression for the graded commutator $[F, G]$ of two graded maps $F, G\in\operatorname{Hgr} K$ reads

$$[F, G]=F\circ G-(-1)^{FG}G\circ F .$$

The overlined symbol of a graded object used as the exponent of (-1) denotes the degree of that object $+1$, mod 2. For instance,

$$(-1)^{\bar{F}}\equiv(-1)^{F+1}\equiv-(-1)^{F} .$$

0. Preliminaries

In this section we recall some notions and results which will be needed. For further details see ref. [11].

Let $\{K, d\}$ be a cochain complex and $F\in\operatorname{Hgr} K$. The map

$$L_F=F\circ d-(-1)^F d\circ F=[F, d]$$

is called the *lievization* of F. Evidently, $L_F\in\operatorname{Hgr} K$ and $\deg L_F=\deg F+1$.

The lievization can be considered as a map,

$$L:\operatorname{Hgr} K\to\operatorname{Hgr} K .$$

It is easy to see that

$$L_{L_F}=0, \quad \text{i.e.} \quad L^2=0, \quad F\in\operatorname{Hgr} K .$$

In other words, $\{\operatorname{Hgr} K, L\}$ is a cochain complex.

For $F, G\in\operatorname{Hgr} K$, we have the following elementary equalities:

$$\begin{aligned}
&(\mathrm{a})\quad L_{F\circ G}=F\circ L_G+(-1)^G L_F\circ G\,,\\
&(\mathrm{b})\quad L_{[F,G]}=[F,L_G]+(-1)^G[L_F,G]\,,\\
&(\mathrm{c})\quad L_{[L_F,G]}=[L_F,L_G],\qquad L_{[F,L_G]}=(-1)^{\bar G}[L_F,L_G]\,.
\end{aligned}\tag{1}$$

Let $\{K,d\}=\{\Lambda^*(M),d\}$. A differential form $\omega\in\Lambda^i(M)$ can be interpreted as a map $\omega\in\mathrm{Hgr}^i K$,

$$\varrho\mapsto\omega\wedge\varrho,\quad \varrho\in\Lambda^*(M)\,.$$

Then $[\omega_1,\omega_2]=0$, $\omega_1,\omega_2\in\Lambda^*(M)$ and

$$L_\omega=(-1)^{\bar\omega}d\omega\,.\tag{2}$$

Also, for arbitrary $\omega,\varrho\in\Lambda^*(M)$ and $X,Y\in\mathscr{D}(M)$ we have

$$\begin{aligned}
&(\mathrm{a})\quad [i_X,i_Y]=0\,,\\
&(\mathrm{b})\quad [i_X,\omega]=X\rfloor\omega \quad\text{and}\quad [\omega,i_X]=(-1)^{\bar\omega}X\rfloor\omega\,,\\
&(\mathrm{c})\quad [i_X,L_Y]=[L_X,i_Y]=i_{[X,Y]}\,,\\
&(\mathrm{d})\quad [L_X,\omega]=X(\omega)\quad\text{and}\quad [\omega,L_X]=-X(\omega)\,,\\
&(\mathrm{e})\quad [i_X,L_\omega]=(-1)^{\bar\omega}X\rfloor d\omega\quad\text{and}\quad [L_\omega,i_X]=-X\rfloor d\omega\,,\\
&(\mathrm{f})\quad [L_X,L_Y]=L_{[X,Y]}\,,\\
&(\mathrm{g})\quad [L_X,L_\omega]=(-1)^{\bar\omega}dX(\omega)\quad\text{and}\quad [L_\omega,L_X]=(-1)^{\omega}dX(\omega)\,.
\end{aligned}\tag{3}$$

We define the *L-commutator* of $F,G\in\mathrm{Hgr}\,K$ as

$$[\![F,G]\!]=\tfrac12[L_F,G]+\tfrac12(-1)^{\bar G}[F,L_G]\,.$$

We will also use the following alternative form of the L-commutator:

$$[\![F,G]\!]=[L_F,G]+L_\Delta\,,$$

where $\Delta=\frac12(-1)^{\bar G}[F,G]$.

The L-commutator has the following properties:

(i) $[\![F,G]\!]=-(-1)^{F\bar G}[\![G,F]\!]$,

(ii) $L_{[\![F,G]\!]}=[L_F,L_G]$ and, in particular, $[\![F,L_G]\!]=\frac12 L_{[\![F,G]\!]}$,

(iii) $(-1)^{F\bar H}[\![F,[\![G,H]\!]]\!]+(-1)^{G\bar H}[\![H,[\![F,G]\!]]\!]+(-1)^{F\bar G}[\![G,[\![H,F]\!]]\!]$
$=\frac13 L_{\mathrm{Flor}(F,G,H)}$,

where

$$\begin{aligned}\mathrm{Flor}(F,G,H)=&(-1)^{F\bar H+\bar G}[F,[\![G,H]\!]]\\
&+(-1)^{H\bar G+\bar F}[H,[\![F,G]\!]]+(-1)^{GF+\bar H}[G,[\![H,F]\!]]\,,\end{aligned}$$

The following formulae are a direct consequence of these properties and of (1)–(3):

$$\begin{aligned}
&(\mathrm{a}) \quad [\![F\circ G, H]\!] = F\circ[\![G,H]\!] + (-1)^{GH}[\![F,H]\!]\circ G \\
&\qquad\qquad + \tfrac{1}{2}\{(-1)^{G}L_F\circ[G,H] + (-1)^{GH}[F,H]\circ L_G\}\,,\\
&(\mathrm{b}) \quad [\![F, G\circ H]\!] = [\![F,G]\!]\circ H + (-1)^{FG}G\circ[\![F,H]\!] \\
&\qquad\qquad + \tfrac{1}{2}\{(-1)^{G+H+1}[F,G]\circ L_H + (-1)^{FG}L_G\circ[F,H]\},
\end{aligned}$$

[(a) and (b) are valid for an arbitrary cochain complex $\{K, d\}$],

$$\begin{aligned}
&(\mathrm{c}) \quad [\![\omega, i_X]\!] = -\tfrac{1}{2}d(X\rfloor\omega) - X\rfloor d\omega,\\
&(\mathrm{d}) \quad [\![\omega, L_X]\!] = -\tfrac{1}{2}L_{X(\omega)},\\
&(\mathrm{e}) \quad [\![\omega, \varrho]\!] = 0\,,\\
&(\mathrm{f}) \quad [\![i_X, i_Y]\!] = i_{[X,Y]}\,,\\
&(\mathrm{g}) \quad [\![\omega, \Theta]\!] = [L_\omega, \Theta] - \tfrac{1}{2}L_{[\omega,\Theta]},\\
&\qquad\; [\![\Theta, \omega]\!] = [\Theta, d\omega] + \tfrac{1}{2}(-1)^{\omega}L_{[\Theta,\omega]}\,, \quad \Theta\in\mathscr{D}_2(M)\,,\\
&(\mathrm{h}) \quad [\![\varrho, L_\Delta]\!] = \tfrac{1}{2}(-1)^{\bar{\varrho}}L_{[d\varrho,\Delta]}\,.
\end{aligned} \tag{4}$$

A graded operator

$$\nabla : \Lambda^*(M) \to \Lambda^*(M)$$

is said to be a *super-differential* of order $\le k$ if

$$[\omega_0, [\omega_1, \cdots[\omega_k, \nabla]\cdots]] = 0\,,$$

for every $\omega_0, \omega_1, \ldots, \omega_k \in \Lambda^*(M)$.

We denote by $\mathrm{S}\mathscr{D}\mathrm{iff}\,M$ the graded algebra of all super-differential operators acting on $\Lambda^*(M)$.

A differential form $\omega\in\Lambda^*(M)$ regarded as an operator,

$$\omega : \Lambda^*(M) \to \Lambda^*(M); \quad \alpha \mapsto \omega\wedge\alpha, \quad \alpha\in\Lambda^*(M)\,,$$

is a zeroth-order super-differential operator.

Also a multi-vector field V can be understood as an operator acting on $\Lambda^*(M)$. Namely, if $V = X_1\wedge\cdots\wedge X_k \in \mathscr{D}_k(M)$, then

$$\varrho \mapsto i_V(\varrho) \equiv V\rfloor\varrho = X_1\rfloor(\cdots\rfloor(X_k\rfloor\varrho))\,, \quad \varrho\in\Lambda^*(M)\,.$$

It follows from (3b) that V is a super-differential operator of order $k-1$. The same conclusion holds for an element

$$\omega\otimes V \in \Lambda^i(M)\otimes\mathscr{D}_j(M)\,,$$

which is understood as the operator

$$\omega\otimes V: \Lambda^*(M)\to\Lambda^*(M);\quad \varrho\mapsto\omega\wedge i_V(\varrho),\quad \varrho\in\Lambda^*(M)\,.$$

In such way we get the imbedding

$$\Lambda^*(M)\otimes\mathscr{D}_*(M)\hookrightarrow \mathrm{S}\mathscr{D}\mathrm{iff}\,M\,.$$

The "sub-imbeddings" $\mathscr{D}_*(M)\hookrightarrow\mathrm{S}\mathscr{D}\mathrm{iff}\,M$ and $\Lambda^*(M)\otimes\mathscr{D}(M)\hookrightarrow\mathrm{S}\mathscr{D}\mathrm{iff}\,M$ of the previous one will be used below in the unification theorem.

Proposition. *The minimal subalgebra of* Hgr $\Lambda^*(M)$ *which is closed with respect to L and contains all insertion operators* i_X, $X\in\mathscr{D}_*(M)$, *and all multiplication operators* ω, $\omega\in\Lambda^*(M)$, *coincides with the algebra of all super-differential operators* $\mathrm{S}\mathscr{D}\mathrm{iff}\,M$. □

The algebra $\mathrm{S}\mathscr{D}\mathrm{iff}\,M$ is closed with respect to L and, therefore, with respect to the L-commutator operation. It follows from property (ii) of the L-commutator that $[\![\mathrm{S}\mathscr{D}\mathrm{iff}\,M, L(\mathrm{S}\mathscr{D}\mathrm{iff}\,M)]\!]\subset L(\mathrm{S}\mathscr{D}\mathrm{iff}\,M)$. Hence, the L-commutator induces a bracket operation on the quotient

$$\mathscr{N}(M)=\mathrm{S}\mathscr{D}\mathrm{iff}\,M/L(\mathrm{S}\mathscr{D}\mathrm{iff}\,M)\,.$$

The following assertion is a direct consequence of properties (i)–(iii) of the L-commutator.

Proposition. *The quotient* $\mathscr{N}(M)$ *equipped with the bracket operation induced from the L-commutator is a graded Lie algebra.* □

Unification theorem. *The compositions*

$$\mathscr{D}_*(M)\hookrightarrow\mathrm{S}\mathscr{D}\mathrm{iff}\,M\to\mathscr{N}(M)\,,$$

$$\Lambda^*(M)\otimes\mathscr{D}(M)\hookrightarrow\mathrm{S}\mathscr{D}\mathrm{iff}\,M\to\mathscr{N}(M),$$

are imbeddings of the graded Lie algebras, supposing that $\mathscr{D}_*(M)$ *is equipped with the Schouten–Nijenhuis bracket and* $\Lambda^*(M)\otimes\mathscr{D}(M)$ *with the Frölicher–Nijenhuis bracket.*

1. Extended Poisson brackets

Let M be a Poisson manifold and $\{\cdot,\cdot\}$ denote the corresponding Poisson bracket. This bracket is defined on the algebra $\mathrm{C}^\infty(M)$, which is the zero-graded part of the graded space of all co-exact forms $\mathscr{P}(M)$: $\mathrm{C}^\infty(M)=\mathscr{P}_0(M)$ (see the section on notation). In this section we extend this bracket from $\mathscr{P}_0(M)$ to the whole space $\mathscr{P}(M)$. This extension equips $\mathscr{P}(M)$ with the structure of a graded Lie algebra (or super-Lie algebra). We carry out the extension procedure in two steps. First, we define the *pre-Poisson bracket* $\langle\cdot,\cdot\rangle$ on the differential form algebra $\Lambda^*(M)$. This bracket does not satisfy the Jacobi identity and is not skew symmetric. However, by making the quotient $\Lambda^*(M)$ with respect to the exact forms $B(M)$, we will obtain the desired bracket on the quotient space $\mathscr{P}(M)=\Lambda^*(M)/B(M)$.

Let $\Theta\in\mathscr{D}_2(M)$ be the bi-vector field on M which gives the considered Poisson structure on M, i.e.,

$$\{f,g\}=\Theta(df,dg)\ .$$

Recall that Θ defines a Poisson structure on M iff $[\![\Theta,\Theta]\!]=0$ (the L-commutator restricted on $\mathscr{D}_*(M)$ coincides with the Schouten–Nijenhuis bracket).

Definition 1. Let Θ be a Poisson structure and ω, $\varrho\in\Lambda^*(M)$. The bracket

$$\langle\omega,\varrho\rangle : \Lambda^*(M)\times\Lambda^*(M)\to\Lambda^*(M)$$

defined by

$$\langle\omega,\varrho\rangle=[\![\omega,[\![\Theta,\varrho]\!]\,]\!]$$

is called the *pre-Poisson bracket* on $\Lambda^*(M)$. □

Evidently this bracket is ℝ-bilinear.

Lemma 2. *We have*

$$[\![\Theta,[\![\mu,\Theta]\!]\,]\!]=-\tfrac{1}{6}L_{\mathrm{Flor}(\mu,\Theta,\Theta)}\ .$$

Proof. It follows directly from property (iii) of the L-commutator, and from the fact that $[\![\Theta,\Theta]\!]=0$. □

Now we need the following assertion proved in ref. [11].

Lemma 3. $B(M)=\Lambda^*(M)\cap L(\mathrm{S}\mathscr{D}\mathrm{iff}\,M)$. □

An immediate consequence of this lemma is that the map $\mathscr{P}(M)\to\mathscr{N}(M)$, induced by the imbedding $\Lambda^*(M)\to\mathrm{S}\mathscr{D}\mathrm{iff}\,M$, is also an imbedding. Below we will consider $\mathscr{P}(M)$ as a subspace of $\mathscr{N}(M)$.

The main properties of the pre-Poisson bracket are given in the next proposition.

Proposition 4. *If* ω, ϱ, $\varphi\in\Lambda^*(M)$, *then*

(1) $\langle\omega,\varrho\rangle=(-1)^{\bar\omega}[d\omega,\,[\Theta,d\varrho]\,]+\frac{1}{2}(-1)^{\varrho}L_{[\omega,[\Theta,d\varrho]]}+\frac{1}{4}(-1)^{\bar\varrho\omega}L_{[[\Theta,\varrho],d\omega]}\,,$

(2) $\langle\omega,\varrho\rangle=-(-1)^{\omega\varrho}\langle\varrho,\omega\rangle+\frac{1}{3}(-1)^{\bar\omega\bar\varrho}L_{\mathrm{Flor}(\omega,\Theta,\varrho)}\,,$

(3) $(-1)^{\omega\varphi}\langle\omega,\langle\varrho,\varphi\rangle\rangle+(-1)^{\varrho\omega}\langle\varrho,\langle\varphi,\omega\rangle\rangle$
$+(-1)^{\varphi\varrho}\langle\varphi,\langle\omega,\varrho\rangle\rangle=d\Omega(\omega,\varrho,\varphi)\,,$

where $\Omega(\omega,\varrho,\varphi)\in\Lambda^*(M)$ is a multi-super-differential operator.

Proof.

(1) By means of the alternative form of the L-commutator $[\![\varrho,\Theta]\!]$ and of (2) and (4h) we have

$$\langle\omega,\varrho\rangle=(-1)^{\varrho}[\![\omega,[\![\varrho,\Theta]\!]\,]\!]=-[\![\omega,\,[d\varrho,\Theta]\,]\!]+\tfrac{1}{2}(-1)^{\bar\varrho+\bar\omega\bar\Delta}L_{[\Delta,d\omega]}\,,$$

where $\Delta=-\frac{1}{2}[\varrho,\Theta]$. Finally we get the desired result by applying the alternative form of the L-commutator $[\![\omega,[d\varrho,\Theta]\,]\!]$.

(2) It follows from property (iii) of the L-commutator and of $[\![\omega,\varrho]\!]=0$ that

$$(-1)^{\bar\omega\bar\varrho}[\![\omega,[\![\Theta,\varrho]\!]\,]\!]+(-1)^{\bar\varrho}[\![\varrho,[\![\omega,\Theta]\!]\,]\!]=\tfrac{1}{3}L_{\mathrm{Flor}(\omega,\Theta,\varrho)}\,.$$

To conclude observe that

$$[\![\varrho,[\![\omega,\Theta]\!]\,]\!]=(-1)^{\omega}[\![\varrho,[\![\Theta,\omega]\!]\,]\!]=(-1)^{\omega}\langle\varrho,\omega\rangle\,.$$

(3) By definition

$$\langle\omega,\langle\varrho,\varphi\rangle\rangle=[\![\omega,[\![\Theta,[\![\varrho,[\![\Theta,\varphi]\!]\,]\!]\,]\!]\,]\!]\,.$$

Then, by applying property (iii) of the L-commutator to $[\![\Theta,[\![\varrho,[\![\Theta,\varphi]\!]\,]\!]\,]\!]$ we get

$$\begin{aligned}\langle\omega,\langle\varrho,\varphi\rangle\rangle=&-(-1)^{\varphi\varrho}[\![\omega,[\![\,[\![\Theta,\varphi]\!],[\![\Theta,\varrho]\!]\,]\!]\,]\!]\\&-(-1)^{\varrho+\bar\varphi}[\![\omega,[\![\varrho,[\![\,[\![\Theta,\varphi]\!],\Theta]\!]\,]\!]\,]\!]+\tfrac{1}{3}(-1)^{\varphi}[\![\omega,L_{\mathrm{Flor}(\Theta,\varrho,[\![\Theta,\varphi]\!])}]\!]\,.\end{aligned}\tag{5}$$

It is now the direct consequence of lemma 2 and (4h) that the two last terms of this expression can be rewritten in the form L_Δ, for some $\Delta\in\mathrm{S}\mathscr{D}\mathrm{iff}\,M$.

By using once more property (iii) of the L-commutator we have

$$\begin{aligned}[\![\omega,[\![\,[\![\Theta,\varphi]\!],[\![\Theta,\varrho]\!]\,]\!]\,]\!]=&-(-1)^{\varrho(\bar\omega+\varphi)}[\![\,[\![\Theta,\varrho]\!],[\![\omega,[\![\Theta,\varphi]\!]\,]\!]\,]\!]\\&-(-1)^{\bar\omega(\varrho+\varphi)}[\![\,[\![\Theta,\varphi]\!],[\![\,[\![\Theta,\varrho]\!],\omega]\!]\,]\!]+\tfrac{1}{3}(-1)^{\bar\omega\varrho}L_{\mathrm{Flor}(\omega,[\![\Theta,\varphi]\!],[\![\Theta,\varrho]\!])}\,.\end{aligned}$$

But

$$\begin{aligned}[\![\,[\![\Theta,\varrho]\!],[\![\omega,[\![\Theta,\varphi]\!]\,]\!]\,]\!]&=[\![\,[\![\Theta,\varrho]\!],\langle\omega,\varphi\rangle\,]\!]\\&=-(-1)^{\varrho(\bar\omega+\varphi)}[\![\,\langle\omega,\varphi\rangle,[\![\Theta,\varrho]\!]\,]\!]\\&=-(-1)^{\varrho(\bar\omega+\varphi)}\langle\langle\omega,\varphi\rangle,\varrho\rangle\,,\end{aligned}$$

and, similarly,

$$[\![[\![\Theta,\varphi]\!], [\![[\![\Theta,\varrho]\!],\omega]\!]]\!] = (-1)^{\omega\varrho+\omega\varphi+\bar{\varrho}\varphi}\langle\langle\omega,\varrho\rangle,\varphi\rangle .$$

It now follows from assertion (2) of this proposition and from (4h) that

$$\langle\langle\omega,\varphi\rangle,\varrho\rangle = (-1)^{\omega\varrho+\varrho\varphi+\omega\varphi}\langle\varrho,\langle\varphi,\omega\rangle\rangle + L_{\nabla_1} ,$$

$$\langle\langle\omega,\varrho\rangle,\varphi\rangle = -(-1)^{\varphi(\omega+\varrho)}\langle\varphi,\langle\omega,\varrho\rangle\rangle + L_{\nabla_2} ,$$

for some $\nabla_1, \nabla_2 \in \mathrm{S}\mathscr{D}\mathrm{iff}\, M$.

Finally, by making use of the above formulae, we can rewrite (5) as

$$\begin{aligned}\langle\langle\omega,\varrho\rangle,\varphi\rangle = &-(-1)^{\omega(\varrho+\varphi)}\langle\varrho,\langle\varphi,\omega\rangle\rangle \\ &-(-1)^{\varphi(\omega+\varrho)}\langle\varphi,\langle\omega,\varrho\rangle\rangle + L_\Delta\end{aligned}$$

for some $\Delta \in \mathrm{S}\mathscr{D}\mathrm{iff}\, M$. The result follows now from lemma 3. □

In order to quotient correctly the pre-Poisson bracket up to a bracket on $\mathscr{P}(M)$ we must show that $\langle\omega, d\varrho\rangle$ and $\langle d\omega,\varrho\rangle$ are exact forms for every $\omega,\varrho\in\Lambda^*(M)$. Because of lemma 3 it is sufficient to show that both of them belong to the image of L.

For $\langle\omega, d\varrho\rangle$ this can be done as follows. From (4g) we see that

$$[\![\Theta, d\varrho]\!] = \tfrac{1}{2}(-1)^{\varrho}L_{[\Theta,d\varrho]} .$$

Then, by applying (4h), we get

$$\langle\omega, d\varrho\rangle = [\![\omega, [\![\Theta, d\varrho]\!]]\!] = \tfrac{1}{2}(-1)^{\varrho}[\![\omega, L_{[\Theta,d\varrho]}]\!] = \tfrac{1}{4}(-1)^{\omega+\varrho}L_{[d\omega,[\Theta,d\varrho]]} .$$

The exactness of the form $\langle d\omega,\varrho\rangle$ is proved similarly.

Below $[\psi]$, $\psi\in\Lambda^*(M)$, denotes the equivalence class of ψ modulo $B(M)$.

The correctness of the following definition follows from the above assertions.

Definition 5. The bracket $\{[\omega],[\varrho]\}\in\mathscr{P}(M)$ defined by

$$\{[\omega],[\varrho]\} = \langle\omega,\varrho\rangle \quad (\mathrm{mod}\, B(M))$$

is called the *generalized Poisson bracket.*

In the following we will write $\{\omega,\varrho\}$ instead of $\{[\omega],[\varrho]\}$.

Now it follows directly from assertions (2) and (3) of proposition 4 and from lemma 3 that the so-defined bracket $\{\cdot,\cdot\}$ on $\mathscr{P}(M)$ is graded skew-symmetric and satisfies the graded Jacobi identity.

This proves our main result:

Theorem 6. *The space $\mathscr{P}(M)$, with the bracket $\{\cdot,\cdot\}$, is a graded Lie algebra.* □

Remark. Let $\Xi\in\mathrm{S}\mathscr{D}\mathrm{iff}\, M$, $\deg\Xi$ be even and $[\![\Xi,\Xi]\!]\in\mathrm{Im}\, L$. Then the bracket

$$\langle \Delta, \nabla \rangle_{\Xi} = [\![\Delta, [\![\Xi, \nabla]\!]\,]\!], \quad \Delta, \nabla \in \mathrm{S}\mathscr{D}\mathrm{iff}\, M\,,$$

defines, by passing to the quotient, a bracket, say $\{\cdot,\cdot\}_{\Xi}$, on $\mathcal{N}(M)$ which turns $\mathcal{N}(M)$ into a graded Lie algebra. This can be proved by the same arguments as above. In particular, every Poisson structure $\Theta \in \mathscr{D}_2(M)$ on M determines the graded Lie algebra structure on $\mathcal{N}(M)$ by means of the bracket $\{\cdot,\cdot\}_{\Theta}$. The restriction of $\{\cdot,\cdot\}_{\Theta}$ to $\mathscr{P}(M) \subset \mathcal{N}(M)$, evidently, coincides with the above introduced Poisson bracket $\{\cdot,\cdot\}$. In other words, $\mathscr{P}(M)$ equipped with the bracket $\{\cdot,\cdot\}$ is a graded Lie subalgebra of $\mathcal{N}(M)$ equipped with the bracket $\{\cdot,\cdot\}_{\Theta}$.

Below we collect some formulae which are necessary for section 2.

Let $\Delta \in N(M) \oplus \Lambda^*(M)$. Then the corresponding decomposition of Δ into a direct sum of two terms looks as

$$\Delta = \Delta_0 \oplus \Delta(1)\,, \tag{6}$$

where $\Delta(1) \in \Lambda^*(M)$ and $\Delta_0 = (\Delta - \Delta(1)) \in N(M)$.

We remark that for $\varphi \in \Lambda^k(M)$,

$$[\Theta, \varphi] = [\Lambda^{k-1}(M) \otimes \mathscr{D}(M)] \oplus \Lambda^{k-2}(M)\,.$$

It is easy to see that

$$[\Theta, \varphi](1) = \Theta \rfloor \varphi\,,$$

and, therefore, the decomposition (6), for $\Delta = [\Theta, \varphi]$, looks as

$$[\Theta, \varphi] = [\Theta, \varphi]_0 + \Theta \rfloor \varphi\,.$$

Moreover, we remark that, since $[\Delta(1), \varphi] = 0$,

$$[\Delta, \varphi] = [\Delta_0, \varphi] = \Delta_0 \rfloor \varphi\,. \tag{7}$$

If $V \in N(M)$ and $\varrho \in \Lambda^*(M)$, then

$$L_V(\varrho) = [V, d](\varrho) = V \rfloor d\varrho - (-1)^V d(V \rfloor \varrho)\,.$$

The following formula stems from the above one for $V = [\Theta, d\omega]_0$, taking into account that, by virtue of (2) and (7),

$$[\Theta, d\omega]_0 \rfloor d\varrho = [[\Theta, d\omega], d\varrho] = (-1)^{\omega\varrho+1} [L_\varrho, [\Theta, d\omega]]\,.$$

Therefore,

$$[L_\varrho, [\Theta, d\omega]] = (-1)^{\omega\varrho+1} L_{[\Theta, d\omega]_0}(\varrho) + (-1)^{\omega\varrho} d([\Theta, d\omega]_0 \rfloor \varrho)\,. \tag{8}$$

We also have

$$L_V(\varphi \wedge \varrho) = L_V(\varphi) \wedge \varrho + (-1)^{V\varphi} \varphi \wedge L_V(\varrho)\,. \tag{9}$$

Lemma 7. *For every $\omega, \varrho \in \Lambda^*(M)$ we have*

$$\langle \varrho, \omega\rangle = (-1)^{\omega\varrho+1} L_{[\Theta, d\omega]_0}(\varrho)$$
$$+ d\{\tfrac{1}{2}(-1)^{\omega\varrho}[\Theta, d\omega]_0 \rfloor \varrho + \tfrac{1}{4}(-1)^{\omega\varrho}[\Theta, \omega]_0 \rfloor d\varrho\} .$$

Proof. By applying (4g) to $[\![\Theta, \omega]\!]$ and then the alternative form of the L-commutator and (4h) we get

$$\langle \varrho, \omega\rangle = [\![\varrho, [\Theta, d\omega]]\!] + \tfrac{1}{2}(-1)^{\omega}[\![\varrho, L_{[\Theta,\omega]}]\!]$$
$$= [L_\varrho, [\Theta, d\omega]] + \tfrac{1}{2}(-1)^{\omega} L_{[\varrho,[\Theta,d\omega]]} + \tfrac{1}{4}(-1)^{\omega+\varrho} L_{[d\varrho,[\Theta,\omega]]} .$$

We now obtain the necessary result by applying (8) to the first term of this expression and (7) and (2) to the second and the third ones. □

Corollary 8.

$$\langle \omega, \varrho\rangle = (-1)^{\omega} L_{[\Theta,d\omega]_0}(\varrho) + dB_\omega(\varrho) ,$$

where

$$B_\omega = -\tfrac{1}{2}[\Theta, d\omega]_0 \rfloor \varrho - \tfrac{1}{4}(-1)^{\omega}[\Theta, \omega]_0 \rfloor d\varrho$$
$$-\tfrac{1}{3}(-1)^{\omega\varrho}\mathrm{Flor}(\omega, \Theta, \varrho) .$$

Proof. We obtain the result by rewriting $\langle \omega, \varrho\rangle$ according to proposition 4 (2) and then applying lemma 7 to $\langle \varrho, \omega\rangle$.

2. Generalized hamiltonian fields

In this section we define the generalized hamiltonian and canonical fields and establish for them the analogues of the standard general formulae.

Throughout this section the symbol "$\approx$" denotes equality $\mathrm{mod}\, B(M)$ or $\mathrm{mod}\, L(\mathrm{S}\mathscr{D}\mathrm{iff}\, M)$.

First of all, we will introduce the notion of a hamiltonian field into our scheme. Our leading principle in doing this is to preserve for generalized hamiltonian fields corresponding to differential forms the well-known classical formula

$$\{f, g\} = X_f(g) , \tag{10}$$

where $f, g \in C^\infty(M)$ and X_f is the hamiltonian field corresponding to the hamiltonian function f. We will reach this goal by adopting the following

Definition 9. The *hamiltonian field* $X_\omega \in N(M)$ corresponding to $\omega \in \Lambda^*(M)$ is defined as

$$X_\omega = (-1)^\omega [\Theta, d\omega]_0 .$$

Evidently, $X_\omega = X_{\omega+d\varrho}$ and, therefore, the field X_ω depends only on the equivalence class $[\omega] \in \mathscr{P}(M)$ of ω. For this reason, it is correct to set, for $\lambda = [\omega]$, $X_\lambda = X_\omega$.

The element $\lambda \in \mathscr{P}(M)$ is called the *hamiltonian* of the hamiltonian field X_λ. By abuse of language, we will also call the form ω a hamiltonian for X_λ. Later we will see that the above definition coincides with the classical one when $f \in C^\infty(M)$.

Inserting the previous definition into corollary 8, we get

$$\langle \omega, \varrho \rangle \approx L_{X_\omega}(\varrho) . \tag{11}$$

It is natural to define the *Lie derivative* $L_X(\lambda)$ of $\lambda \in \mathscr{P}(M)$ along a "field" $X \in N(M)$ by setting

$$L_X(\lambda) = [X(\varrho)], \quad \lambda = [\varrho] .$$

The correctness of this definition is evident. Now we can rewrite (11) in the form

$$\{\mu, \lambda\} = L_{X_\omega}(\lambda) = L_{X_\mu}(\lambda) ,$$

where $\mu = [\omega]$, $\lambda = [\varrho]$, which generalizes (10).

The form (11) of the pre-Poisson bracket allows us to prove the analogue of the classical formula $\{f, gh\} = \{f, g\}h + g\{f, h\}$:

$$\langle \omega, \varphi \wedge \varrho \rangle \approx \langle \omega, \varphi \rangle \wedge \varrho + (-1)^{\omega\varphi} \varphi \wedge \langle \omega, \varrho \rangle .$$

In fact, it follows from (9), that

$$\langle \omega, \varphi \wedge \varrho \rangle \approx L_{X_\omega}(\varphi \wedge \varrho) = L_{X_\omega}(\varphi) \wedge \varrho + (-1)^{\omega\varphi} \varphi \wedge L_{X_\omega}(\varrho) .$$

The set

$$\operatorname{Ham} \Theta = \{X_\omega \in N(M) \mid \omega \in \Lambda^*(M)\}$$

of all hamiltonian fields (with respect to Θ) is, evidently, a linear subspace of $N(M)$. Moreover, it is a graded Lie subalgebra of $N(M)$. To prove this we need the formula

$$X_\omega = [\![\omega, \Theta]\!] + (-1)^\omega \Theta \rfloor d\omega - \tfrac{1}{2} L_{[\Theta, \omega]} , \tag{12}$$

expressing hamiltonian fields in terms of the bracket $[\![\cdot, \cdot]\!]$. It follows immediately from the definition of X_ω, from (2) and from the alternative form of the L-commutator.

Corollary 10.

$$X_\omega \approx [\![\omega, \Theta]\!] + \varphi, \quad \varphi \in \Lambda^*(M) . \qquad \square$$

We also remark that $[\![Y,\omega]\!]$ is a differential form, when $Y\in N(M)$ and $\omega\in\Lambda^*(M)$. This can be seen from the formula

$$[\![Y,\omega]\!]=Y\rfloor d\omega-\tfrac{1}{2}(-1)^{Y}d(Y\rfloor\omega)\ , \tag{13}$$

which follows straightforwardly from the definition of the bracket $[\![\cdot,\cdot]\!]$.

We are now ready to prove the following analogue of the classical formula $[X_f,X_g]=X_{\{f,g\}}$ (here $[\cdot,\cdot]$ stands for the Lie bracket).

Proposition 11.

$$[\![X_\omega,X_\varrho]\!]\approx X_{\{\omega,\varrho\}}\ .$$

Proof. According to corollary 10, we have $X_\omega\approx[\![\omega,\Theta]\!]+\varphi$, $X_\varrho\approx[\![\varrho,\Theta]\!]+\psi$, $\varphi,\psi\in\Lambda^*(M)$. Therefore,

$$\begin{aligned}[\![X_\omega,X_\varrho]\!]&\approx[\![\,[\![\omega,\Theta]\!]+\varphi,[\![\varrho,\Theta]\!]+\psi]\!]\\&=[\![\,[\![\omega,\Theta]\!],[\![\varrho,\Theta]\!]\,]\!]+[\![\,[\![\omega,\Theta]\!],\psi]\!]+[\![\varphi,[\![\varrho,\Theta]\!]\,]\!]\ .\end{aligned}$$

We note that $[\![\,[\![\omega,\Theta]\!],\psi]\!]$ and $[\![\varphi,[\![\varrho,\Theta]\!]\,]\!]$ are differential forms. But, from property (iii) of the L-commutator, we have

$$\begin{aligned}[\![\,[\![\omega,\Theta]\!],[\![\varrho,\Theta]\!]\,]\!]&+(-1)^{\omega+\varrho}[\![\Theta,[\![\,[\![\omega,\Theta]\!],\varrho]\!]\,]\!]\\&+(-1)^{\omega\varrho}[\![\varrho,[\![\Theta,[\![\omega,\Theta]\!]\,]\!]\,]\!]\approx0\ .\end{aligned}$$

The third term of the above expression is ≈0 (in virtue of lemma 2) and, therefore,

$$\begin{aligned}[\![\,[\![\omega,\Theta]\!],[\![\varrho,\Theta]\!]\,]\!]&=(-1)^{\omega+\varrho}[\![\Theta,[\![\,[\![\omega,\Theta]\!],\varrho]\!]\,]\!]\\&=[\![\{\omega,\varrho\},\Theta]\!]\approx X_{\{\omega,\varrho\}}+\zeta\ ,\end{aligned}$$

where $\zeta\in\Lambda^*(M)$. □

Corollary 12. Ham Θ *is a Lie subalgebra of* $N(M)$. □

Definition 13. A field $X\in N(M)$ is said to be a *canonical field* with respect to a Poisson structure Θ, if

$$[\![X,\Theta]\!]\approx0\ .$$ □

It easy to see that this notion coincides with the classical one if $X\in\mathscr{D}(M)$. The set

$$\mathrm{Can}\,\Theta=\{X\in N(M)\mid[\![X,\Theta]\!]\in N(M)+\mathrm{im}\,L\}$$

of all canonical fields with respect to Θ is, evidently, a linear subspace of $N(M)$.

As in the classical situation, we have

Proposition 14. *The set* Can Θ *is a Lie subalgebra of* $N(M)$.

Proof. It follows from property (iii) of the L-commutator that $[\![[\![X, Y]\!], \Theta]\!] \in N(M) + \operatorname{im} L$ if $[\![X, \Theta]\!]$, $[\![Y, \Theta]\!] \in N(M) + \operatorname{im} L$. □

Finally, we have

Proposition 15.
(1) Ham $\Theta \subset$ Can Θ;
(2) Ham Θ *is an ideal of the graded Lie algebra* Can Θ.

Proof.
(1) For $X_\omega \in$ Ham Θ, we have in virtue of (12)

$$\begin{aligned}[\![X_\omega, \Theta]\!] &= [\![[\![\omega, \Theta]\!] + \varphi - L_\Delta, \Theta]\!] \\ &= [\![[\![\omega, \Theta]\!], \Theta]\!] + [\![\varphi, \Theta]\!] - [\![L_\Delta, \Theta]\!] ,\end{aligned}$$

where $\varphi = (-1)^\omega \Theta \rfloor d\omega$, and $\Delta = -\frac{1}{2}[\Theta, \omega]$. We note that

$$[\![[\![\omega, \Theta]\!], \Theta]\!] \approx 0, \qquad [\![\varphi, \Theta]\!] \approx Y + \psi \in N(M) \oplus \Lambda^*(M), \qquad [\![L_\Delta, \Theta]\!] \approx 0 .$$

Hence,

$$[\![X_\omega, \Theta]\!] = Y + \psi + L_\nabla, \quad \text{for some } \nabla \in \mathrm{S}\mathscr{D}\mathrm{iff}\, M ,$$

and, therefore, $[\![X_\omega, \Theta]\!](1) = \psi + d\nabla(1)$. On the other hand, $[\![X_\omega, \Theta]\!](1) = 0$. Therefore, $\psi \in \operatorname{Im} d$ and $[\![X_\omega, \Theta]\!] \approx Y$.

(2) With the same notations as above, we have for $X_\omega \in$ Ham Θ and $Y \in$ Can Θ

$$\begin{aligned}[\![Y, X_\omega]\!] &= [\![Y, [\![\omega, \Theta]\!] + \varphi - L_\Delta]\!] \\ &= [\![Y, [\![\omega, \Theta]\!]]\!] + [\![Y, \varphi]\!] - [\![Y, L_\Delta]\!] \\ &= -(-1)^{\Theta(\omega+Y)}[\![\Theta, [\![Y, \omega]\!]]\!] - (-1)^{Y(\Theta+\omega)}[\![\omega, [\![\Theta, Y]\!]]\!] \\ &\quad + \tfrac{1}{3} L_{\mathrm{Flor}(Y,\omega,\Theta)} + [\![Y, \varphi]\!] - [\![Y, L_\Delta]\!]\end{aligned}$$

(because of property (iii) of the L-commutator). We note that

$$-(-1)^{Y(\Theta+\omega)}[\![\omega, [\![\Theta, Y]\!]]\!] + [\![Y, \varphi]\!] \approx \psi \in \Lambda^*(M), \qquad [\![Y, L_\Delta]\!] \approx 0 .$$

Moreover, because of (13) we have

$$\begin{aligned}-(-1)^{\Theta(\omega+Y)}[\![\Theta, [\![Y, \omega]\!]]\!] &= [\![[\![Y, \omega]\!], \Theta]\!] \\ &= [\![(Y \rfloor d\omega) - \tfrac{1}{2}(-1)^Y d(Y \rfloor \omega), \Theta]\!] \approx [\![Y \rfloor d\omega, \Theta]\!] .\end{aligned}$$

Then,

$$[\![Y, X_\omega]\!] \approx [\![Y \rfloor d\omega, \Theta]\!] + \psi ,$$

or, by making use of (12),

$$[\![Y, X_\omega]\!] \approx X_{(Y \rfloor d\omega)} + \varrho, \quad \text{for some } \varrho \in \Lambda^*(M) . \tag{14}$$

But, on the other hand,

$$[\![Y, X_\omega]\!] = Z + L_{\bar{v}} , \tag{15}$$

where $Z \in N(M)$, as follows from the unification theorem. Therefore, comparing (14) and (15), we can conclude that

$$[\![Y, X_\omega]\!] \approx X_{(Y \rfloor d\omega)} . \qquad \square$$

Corollary 16. *The quotient* Can Θ/Ham Θ *is a graded Lie algebra.* □

Proposition 17.

$$[\![Y, X_\omega]\!] \approx X_{(Y \rfloor d\omega)} \approx X_{Y(\omega)} .$$

Proof. The first equality has already been proved before and the second one follows from

$$Y(\omega) = Y \rfloor d\omega + d(Y \rfloor \omega) . \qquad \square$$

Finally, applying the unification theorem, we have

Corollary 18. *The Frölicher–Nijenhuis bracket of Y and X_ω is the hamiltonian field $X_{(Y \rfloor d\omega)}$.* □

Concluding this section we show that, if M is a symplectic manifold, then our generalized Poisson bracket coincides with the bracket defined by Michor [7]. In fact, if $\Theta = \sum_i \partial/\partial x_i \wedge \partial/\partial p_i$, where (x_i, p_i) are canonical coordinates, and $\omega = f_0 df_1 \wedge \cdots \wedge df_k$, then

$$X_\omega = \sum_{s=0}^{k} (-1)^s df_0 \wedge df_1 \wedge \cdots \wedge \hat{df_s} \wedge \cdots \wedge df_k X_{f_s} , \tag{16}$$

where

$$X_f = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial p_i} \right)$$

is the standard hamiltonian field corresponding to $f \in C^\infty(M)$. It remains to observe that (16) coincides with the formula of Michor for generalized hamiltonian fields.

3. The "differential" of the Poisson bracket

A Poisson structure on a manifold M provides in a natural way the algebra $\Lambda^*(M)$ with a graded Lie algebra structure, as was found for the first time by Karasev [9] and by Kosmann-Schwarzbach and Magri [10]. The corresponding bracket, say $]\cdot,\cdot[$, if supposed to be a graded derivation with respect to each of its arguments, is defined uniquely by the "initial data" conditions

$$]f, g[=0, \qquad]f, dg[=\{f, g\}, \qquad]df, dg[=d\{f, g\}, \quad f, g\in C^\infty(M)\ , \qquad (17)$$

where $\{\cdot,\cdot\}$ is the given Poisson bracket.

We see from (17) that the bracket $]\cdot,\cdot[$ is the "differential" of the Poisson bracket $\{\cdot,\cdot\}$, at least concerning functions on M. Below we extend this result to all co-exact forms by showing that the map

$$\mathscr{P}(M)=\Lambda^*(M)/B(M)\to\Lambda^*(M), \quad \mathscr{P}(M)\ni[\omega]\mapsto d\omega\in\Lambda^*(M)\ ,$$

is a homomorphism of the graded Lie algebras, where $\mathscr{P}(M)$ is supplied with the generalized Poisson bracket of section 1 and $\Lambda^*(M)$ with the bracket $]\cdot,\cdot[$.

It was observed by Krasil'shchik [8] that the "unification" techniques (see ref. [11] or section 0) allows one to define the bracket $]\cdot,\cdot[$ as the "Schouten–Nijenhuis bracket" with respect to the hamiltonian complex associated with the considered Poisson structure. We start by recalling Krasil'shchik's definition.

First, we note that a given Poisson structure $\Theta\in\mathscr{D}_2(M)$ on M provides the graded commutative algebra $\mathscr{D}_*(M)=\sum_i\mathscr{D}_i(M)$ of multi-vector fields on M with the differential

$$d_\Theta:\mathscr{D}_i(M)\to\mathscr{D}_{i+1}(M), \quad i=0, 1, \dots ,$$

where $d_\Theta(\Xi)=[\![\Theta,\Xi]\!]$ and $[\![\cdot,\cdot]\!]$ stands for the standard Schouten–Nijenhuis bracket [recall that, in view of the unification theorem, this coincides with the L-commutator restricted to $\mathscr{D}_*(M)$]. Taking $\{K, d\}=\{\mathscr{D}_*(M), d_\Theta\}$ to be the basic complex for the "unification" construction (see section 0) one can supply the space Hgr $\mathscr{D}_*(M)$ with the L-commutator operation, which will be denoted by $[\![\cdot,\cdot]\!]_\Theta$. So we have, by definition,

$$[\![F, G]\!]=\tfrac{1}{2}[L_F^\Theta, G]'+\tfrac{1}{2}(-1)^G[F, L_G^\Theta]'\ ,$$

where $F, G\in\mathrm{Hgr}\,\mathscr{D}_*(M)$ and $L_F^\Theta=[d_\Theta, F]'$ is the lievization of F with respect to d_Θ. Here we use $[\cdot,\cdot]'$ for the graded commutator in Hgr $\mathscr{D}_*(M)$ to distinguish it from the graded commutator $[\cdot,\cdot]$ in Hgr $\Lambda^*(M)$.

Second, we observe that differential forms on M can be interpreted naturally as elements of Hgr $\mathscr{D}_*(M)$. Namely, a given $\omega\in\Lambda^*(M)$ defines the graded map $V\mapsto\omega\rfloor V$, $V\in\mathscr{D}_*(M)$, of $\mathscr{D}_*(M)$ into itself where "$\rfloor$" denotes the insertion operation of differential forms into multi-vector fields. This interpretation gives us the possibility to apply the L-commutator operation $[\![\cdot,\cdot]\!]_\Theta$ to differential forms. In such a way we get the desired extension. The necessary details are given below.

Our next step is to deduce some explicit formulae for the operators L^{Θ}_{ω}, $\omega \in \Lambda^*(M)$. To do this we need the following identities, which result directly from (4a) and (4b):

$$
\begin{aligned}
\llbracket [F,G],H\rrbracket &= [F,\llbracket G,H\rrbracket] + (-1)^{G\bar{H}}[\llbracket F,H\rrbracket,G] \\
&\quad + \tfrac{1}{2}(-1)^{G}[L_F,[G,H]] + \tfrac{1}{2}(-1)^{\bar{G}H}[[F,H],L_G]\,, \\
\llbracket F,[G,H]\rrbracket &= [\llbracket F,G\rrbracket,H] + (-1)^{\bar{F}G}[G,\llbracket F,H\rrbracket] \\
&\quad + \tfrac{1}{2}(-1)^{G+H+1}[[F,G],L_H] + \tfrac{1}{2}(-1)^{\bar{F}\bar{G}}[L_G,[F,H]]\,.
\end{aligned}
\tag{18}
$$

The first result we need is

Proposition 19. *We have for $f \in C^{\infty}(M)$*

(1) *$L^{\Theta}_f = -X_f$, where the vector field X_f is regarded as the multiplication operator $V \mapsto X_f \wedge V$ on $\mathcal{D}_*(M)$;*

(2) *$L^{\Theta}_{df} = L_{X_f}$.*

Proof.

(1) Let $V \in \mathcal{D}_*(M)$. Then applying the definition of lievization and formula (4b) we get

$$
\begin{aligned}
L^{\Theta}_f &= [f,d_\Theta]'(V) = f\llbracket\Theta,V\rrbracket - \llbracket\Theta,fV\rrbracket \\
&= f\llbracket\Theta,V\rrbracket - \llbracket\Theta,f\rrbracket \wedge V - f\llbracket\Theta,V\rrbracket \\
&= -\llbracket\Theta,f\rrbracket \wedge V = -X_f \wedge V\,.
\end{aligned}
$$

(2) By definition

$$
L^{\Theta}_{df}(V) = [df,d_\Theta]'(V) = df \rfloor \llbracket\Theta,V\rrbracket + \llbracket\Theta, df\rfloor V\rrbracket\,.
$$

Next, we note that $\omega \rfloor V = -[\omega,V]$ for $\omega \in \Lambda^1(M)$. Therefore,

$$
L^{\Theta}_{df}(V) = -[df,\llbracket\Theta,V\rrbracket] - \llbracket\Theta,[df,V]\rrbracket\,. \tag{19}
$$

Now the second of formulae (18) gives us the following expression for the second term of (19):

$$
\begin{aligned}
\llbracket\Theta,[df,V]\rrbracket &= [\llbracket\Theta,df\rrbracket,V] - [df,\llbracket\Theta,V\rrbracket] \\
&\quad + \tfrac{1}{2}(-1)^{V}[[\Theta,df],L_V] + \tfrac{1}{2}[L_{df},[\Theta,V]]\,.
\end{aligned}
$$

Observing that $[L_{df},[\Theta,V]] = 0$ and substituting the last expression into (19), we get

$$
L^{\Theta}_{df}(V) = -[\llbracket\Theta,df\rrbracket,V] + \tfrac{1}{2}(-1)^{\bar{V}}[[\Theta,df],L_V]\,. \tag{20}
$$

But from (4g) and (12) one can see that

$$[\![\Theta, df]\!] = -\tfrac{1}{2}L_{[\Theta,df]} = -\tfrac{1}{2}L_{[\![\Theta,f]\!]} = -\tfrac{1}{2}L_{X_f},$$
$$[\Theta, df] = [\![\Theta, f]\!] = X_f.$$

This allows us to rewrite (20) as

$$L^{\Theta}_{df}(V) = \tfrac{1}{2}[L_{X_f}, V] + \tfrac{1}{2}(-1)^p[X_f, L_V] = [\![X_f, V]\!] = L_{X_f}(V).$$

So

$$L^{\Theta}_{df} = L_{X_f}. \qquad \square$$

To go on we need the following result.

Lemma 20. *For any $X \in \mathscr{D}(M)$, $\omega \in \Lambda^*(M)$ the formula*

$$L_X(\omega \rfloor V) = -X(\omega)\rfloor V + \omega \rfloor L_X(V) \tag{21}$$

holds.

Proof. Formula (21) is the infinitesimal version of the naturality property of the insertion operation. In fact, if $F: M \to N$ is a diffeomorphism, then

$$F(\omega\rfloor V) = ((F^{-1})^*(\omega))\rfloor F(V),$$

where $F(W)$ denotes the image of $W \in \mathscr{D}_*(M)$ along F. In particular, if $X \in \mathscr{D}(M)$ and $A_t: M \to M$ is the one-parameter group of diffeomorphisms generated by X, then

$$A_t(\omega\rfloor V) = (A^*_{-t}(\omega))\rfloor A_t(V). \tag{22}$$

But

$$\frac{d}{dt}A_t(W)|_{t=0} = L_X(W), \qquad \frac{d}{dt}A^*_{-t}(\omega)|_{t=0} = -X(\omega).$$

So, differentiating (22) with respect to t, we obtain (21). $\square$

It follows directly from (21) that

$$[L_X, \omega]' = -X(\omega). \tag{23}$$

Now we are ready to compute the bracket $[\![\cdot,\cdot]\!]_\Theta$ for the simplest arguments.

Proposition 21. *If $f, g \in C^\infty(M)$, then*

$$[\![f, g]\!]_\Theta = 0, \qquad [\![f, dg]\!]_\Theta = \{f, g\}, \qquad [\![df, dg]\!]_\Theta = d\{f, g\}.$$

Proof. First, by virtue of proposition 19, we have

$$\begin{aligned}[\![f,g]\!]_\Theta &= \tfrac{1}{2}[L_f^\Theta, g]' - \tfrac{1}{2}[f, L_g^\Theta]' \\ &= -\tfrac{1}{2}[X_f, g]' + \tfrac{1}{2}[f, X_g]' = 0 .\end{aligned}$$

Similarly, but taking into account (23) in addition, we get

$$\begin{aligned}[\![f,dg]\!]_\Theta &= \tfrac{1}{2}[L_f^\Theta, dg]' + \tfrac{1}{2}[f, L_{dg}^\Theta]' = -\tfrac{1}{2}[X_f, dg]' + \tfrac{1}{2}[f, L_{X_g}]' \\ &= \tfrac{1}{2}X_f(g) - \tfrac{1}{2}X_g(f) = \tfrac{1}{2}\{f,g\} - \tfrac{1}{2}\{g,f\} = \{f,g\} ,\end{aligned}$$

and, finally,

$$\begin{aligned}[\![df,dg]\!]_\Theta &= \tfrac{1}{2}[L_{df}^\Theta, dg]' + \tfrac{1}{2}[df, L_{dg}^\Theta]' = -\tfrac{1}{2}[L_{X_f}, dg]' - \tfrac{1}{2}[df, L_{X_g}]' \\ &= \tfrac{1}{2}X_f(dg) - \tfrac{1}{2}X_g(df) = \tfrac{1}{2}d(X_f(g) - X_g(f)) = d\{f,g\} . \qquad \square\end{aligned}$$

The previous proposition supplies us with the necessary "initial data" to prove the following central formula:

$$[\![\omega, \varrho]\!]_\Theta = (-1)^{\omega}[d\omega, [\Theta, \varrho]] + (-1)^{\omega} d[\omega, [\Theta, \varrho]] - [\omega, [\Theta, d\varrho]] . \tag{24}$$

This is done as follows.

Denote by $]\!]\omega, \varrho[\![$ the right hand side of (24). Then we have

Lemma 22. *If $f, g \in C^\infty(M)$, then*

$$]\!]f,g[\![= 0, \qquad]\!]f, dg[\![= \{f,g\}, \qquad]\!]df, dg[\![= d\{f,g\} .$$

Proof. It is evident that $]\!]f,g[\![= 0$. Next, from the definition we see that $]\!]f,dg[\![= -[df, [\Theta, dg]]$. But by virtue of proposition 4(1) the last expression coincides with $\{f,g\}$. Similarly, $]\!]df,dg[\![= -d[df, [\Theta, dg]] = d\{f,g\}$ [proposition 4(1)]. □

Next, we have

Proposition 23. *Both brackets $[\![\cdot,\cdot]\!]_\Theta$ and $]\!]\cdot,\cdot[\![$ are commutative (in the "graded" sense) and satisfy the following conditions:*

$$[\![\omega \wedge \varrho, \lambda]\!]_\Theta = \omega \wedge [\![\varrho, \lambda]\!]_\Theta + (-1)^{\varrho\lambda}[\![\omega, \lambda]\!]_\Theta \wedge \varrho , \tag{25}$$

$$]\!]\omega \wedge \varrho, \lambda[\![= \omega \wedge]\!]\varrho, \lambda[\![+ (-1)^{\varrho\lambda}]\!]\omega, \lambda[\![\wedge \varrho . \tag{26}$$

Proof. The bracket $[\![\cdot,\cdot]\!]_\Theta$ is commutative by being the L-commutator. Commutativity of $]\!]\cdot,\cdot[\![$ results from the Jacobi identity for the graded commutator $[\cdot,\cdot]$ applied to each of the three terms constituting the right hand side of (24).

To prove (25) we note that compositions of differential forms regarded as insertion operators on $\mathcal{D}_*(M)$ coincide with the exterior multiplication operation. Then (25) follows from (4a), the last two terms of which vanish because of the graded commutativity of differential forms interpreted as insertion operators.

Finally, (26) is checked easily by direct computation. □

Now, we are ready to prove the main result of this section.

Proposition 24. *Formula* (24) *holds.*

Proof. In other words, it suffices to prove that the brackets $[\![\cdot,\cdot]\!]_\Theta$ and $]\!]\cdot,\cdot[\![$ on $\Lambda^*(M)$ coincide. But proposition 21 and lemma 22 show that they actually coincide on functions and their differentials. On the other hand, it follows from proposition 23 that each of these brackets for arbitrary differential forms is reduced to the corresponding brackets of functions and their differentials exactly in the same way. □

We denote by d also the map $\mathcal{P}(M)\to\Lambda^*(M)$, which is given by $\mathcal{P}(M)\ni[\omega]\mapsto d\omega\in\Lambda^*(M)$. Then we have

Theorem 25. The map $d:\mathcal{P}(M)\to\Lambda^*(M)$ is a homomorphism of graded Lie algebras, where $\mathcal{P}(M)$ is supposed to be equipped with the generalized Poisson bracket $\{\cdot,\cdot\}$ and $\Lambda^*(M)$ with the bracket $[\![\cdot,\cdot]\!]_\Theta$.

Proof. It follows from (24) that

$$[\![d\omega, d\varrho]\!]_\Theta=(-1)^{\omega}d[d\omega,[\Theta,d\varrho]]\,.$$

Also, we see from proposition 4(1) that

$$d\{\omega,\varrho\}=(-1)^{\omega}d[d\omega,[\Theta,d\varrho]]$$

because $[\omega,[\Theta,d\varrho]]$ and $[[\Theta,\varrho],d\omega]$ are differential forms and, therefore,

$$L_{[\omega,[\Theta,d\varrho]]}=\pm d[\omega,[\Theta,d\varrho]],\qquad L_{[[\Theta,\varrho],d\omega]}=\pm d[[\Theta,\varrho],d\omega]\,.$$ □

We conclude this section with the formula

$$\{\omega,\varrho\}=[\![\omega,d\varrho]\!]_\Theta \bmod B(M)\,, \tag{27}$$

which results directly from proposition 4(1) and (24). This shows that the Poisson bracket on $\mathcal{P}(M)$ is determined completely by its "differential", i.e. by the bracket $[\![\cdot,\cdot]\!]_\Theta$.

4. The graded extension of the hamiltonian map

Let $\Theta\in\mathscr{D}_2(M)$ be a Poisson structure on M. The map $d_\Theta : \mathrm{C}^\infty(M)\to\mathscr{D}(M)$,

$$d_\Theta f\equiv[\![\Theta,f]\!]\equiv X_f\,,$$

is a differentiation of the algebra $\mathrm{C}^\infty(M)$ with values in the $\mathrm{C}^\infty(M)$-module $\mathscr{D}(M)$, i.e.,

$$d_\Theta(fg)=fd_\Theta(g)+gd_\Theta(f)\ .$$

The universality of the differentiation $d : \mathrm{C}^\infty(M)\to\Lambda^1(M)$ yields the unique homomorphism of $\mathrm{C}^\infty(M)$-modules $\Gamma_\Theta : \Lambda^1(M)\to\mathscr{D}(M)$ which makes the following diagram commutative:

$$\begin{tikzcd}
\Lambda^1(M) \arrow[rr, "\Gamma_\Theta"] & & \mathscr{D}(M) \\
& \mathrm{C}^\infty(M) \arrow[ul, "d"] \arrow[ur, "d_\Theta"'] &
\end{tikzcd}\ .$$

In particular, $\Gamma_\Theta(fdg)=fX_g$. Γ_Θ is called the *hamiltonian map* corresponding to the Poisson structure Θ. Regarding the exterior algebra $\Lambda^*(M)$ to be the graded (or "super") extension of the algebra $\mathrm{C}^\infty(M)$, it is natural to ask for the graded extension of this map. The following proposition, which is evident, answers this question.

Proposition 26. *There exists a unique homomorphism of graded commutative algebras*

$$\Gamma_\Theta : \Lambda^*(M)\to\mathscr{D}_*(M)\ ,$$

such that for $f\in\mathrm{C}^\infty(M)$
(1) $\Gamma_\Theta(f)=f$,
(2) $\Gamma_\Theta(df)=d_\Theta f=X_f$. □

In particular, for $f_0, f_1, \ldots, f_k\in\mathrm{C}^\infty(M)$, we have

$$\Gamma_\Theta(f_0df_1\wedge\cdots\wedge df_k)=f_0X_{f_1}\wedge\cdots\wedge X_{f_k}\ .$$

The main properties of this extended hamiltonian map are the following:

Proposition 27.
(1) $\Gamma_\Theta\circ d=d_\Theta\circ\Gamma_\Theta$, *i.e., the extended hamiltonian map is a cochain map of the de Rham complex $\{\Lambda^*(M), d\}$ into the Hamilton complex $\{\mathscr{D}_*(M), d_\Theta\}$.*
(2) *For every $\omega, \varrho\in\Lambda^*(M)$,*

$$\Gamma_\Theta([\![\omega,\varrho]\!]_\Theta)=[\![\Gamma_\Theta\omega,\Gamma_\Theta\varrho]\!]\ ,$$

i.e., Γ_Θ is a homomorphism of graded Lie algebras.

Proof.

(1) It is well known and results directly from (4b) that

$$[\![\Theta, V\wedge W]\!]=[\![\Theta, V]\!]\wedge W+(-1)^{V}V\wedge[\![\Theta, W]\!]$$

for every $V, W\in\mathscr{D}_*(M)$. In other words, we have

$$d_\Theta(V\wedge W)=d_\Theta V\wedge W+(-1)^{V}V\wedge d_\Theta W\,. \tag{28}$$

This shows d_Θ to be a differentiation of the graded commutative algebra $\mathscr{D}_*(M)$ as well as d with respect to $\Lambda^*(M)$. So, the mappings $\Gamma_\Theta\circ d$ and $d\circ\Gamma_\Theta$ coincide iff they coincide on $C^\infty(M)$. But this is exactly property 2 of Γ_Θ given in proposition 26.

(2) As before, it is well known, and results directly from (4a) and (4b), that

$$[\![V\wedge W, Z]\!]=V\wedge[\![W, Z]\!]+(-1)^{WZ}[\![V, Z]\!]\wedge W\,,$$

$$[\![V, W\wedge Z]\!]=[\![V, W]\!]\wedge Z+(-1)^{\bar{V}W}W\wedge[\![V, Z]\!]\,.$$

These properties of the Schouten–Nijenhuis bracket and the similar properties of the bracket $[\![\cdot,\cdot]\!]_\Theta$ leave the problem to be checked for differential forms ω, ϱ of degree ≤ 1 only. But in this case the result follows from proposition 21 and from the evident equalities

$$[\![f, g]\!]=0,\qquad [\![f, X_g]\!]=-X_g(f)=\{f, g\}\,,$$

$$\Gamma_\Theta([\![df, dg]\!]_\Theta)=\Gamma_\Theta(d\{f, g\})=d_\Theta\{f, g\}$$

$$=X_{\{f,g\}}=[\![X_f, X_g]\!]=[\![d_\Theta f, d_\Theta g]\!]\,,$$

where $f, g\in C^\infty(M)$. □

Corollary 28. *The hamiltonian map Γ_Θ induces the homomorphism of the cohomology algebras*

$$\Gamma^*_\Theta : H^*(M)\to H^*_\Theta(M)\,,$$

where $H^*_\Theta(M)=\sum_{i\geq 0} H^i_\Theta(M)$ denotes cohomologies of the hamiltonian complex $\{\mathscr{D}_*(M), d_\Theta\}$. □

5. The Poisson "integral" of the Schouten–Nijenhuis bracket

Let $\mathscr{B}_\Theta(M)=d_\Theta(\mathscr{D}_*(M))$ be the space of multi-vector fields on M which are exact with respect to Θ. Then $\Pi_\Theta(M)=\mathscr{D}_*(M)/\mathscr{B}_\Theta(M)$ is the space of multi-vector fields on M which are all co-exact with respect to Θ. As before, we define the map d_Θ as

$$d_\Theta : \Pi_\Theta(M)\to\mathscr{D}_*(M),\quad d_\Theta([V])=d_\Theta V\,,$$

where $[V]=V(\mathrm{mod}\ \mathscr{B}_\Theta(M))$. Now, the diagram

$$\begin{array}{ccc} \Lambda^*(M) & \xrightarrow{\Gamma_\Theta} & \mathscr{D}_*(M) \\ d\uparrow & & \uparrow d_\Theta \\ \mathscr{P}(M) & \xrightarrow[\gamma_\Theta]{} & \Pi_\Theta(M) \end{array} \tag{29}$$

appeals to be completed with a graded Lie algebra structure on $\Pi_\Theta(M)$ and a graded Lie algebra homomorphism γ_Θ which makes it commutative.

We copy formula (27) to define the desired bracket, say, $\{\cdot,\cdot\}_\Theta$ on $\Pi_\Theta(M)$:

$$\{[V],[W]\}_\Theta=[\![V,d_\Theta W]\!] \quad \mathrm{mod}\ \mathscr{B}_\Theta(M)\,, \tag{30}$$

where $V, W\in\mathscr{D}_*(M)$.

Lemma 29. *Definition* (30) *is correct.*

Proof. The Jacobi identity of the Schouten–Nijenhuis bracket applied to Θ, V, $W\in\mathscr{D}_*(M)$ can be rewritten in the form

$$d_\Theta[\![V,W]\!]=[\![d_\Theta V,W]\!]+(-1)^V[\![V,d_\Theta W]\!]\,. \tag{31}$$

In particular, for $W=d_\Theta Z$ we have

$$d_\Theta[\![V,d_\Theta Z]\!]=[\![d_\Theta V,d_\Theta Z]\!]\,. \tag{32}$$

Now, it is seen from (32) that $[\![V,d_\Theta W]\!]\in\mathscr{B}_\Theta(M)$ if either V or W belongs to $\mathscr{B}_\Theta(M)$. □

Because of this lemma we can write $\{V,W\}_\Theta$ instead of $\{[V],[W]\}_\Theta$. Also, below we will make use of the abbreviation $V\backsim W$ instead of $V\equiv W \ \mathrm{mod}\ \mathscr{B}_\Theta(M)$.

Proposition 30. *The bracket* $\{\cdot,\cdot\}_\Theta$ supplies $\Pi_\Theta(M)$ *with a graded Lie algebra structure, i.e., for every* $V, W, Z\in\mathscr{D}_*(M)$ *we have*:

(1) $\{V,W\}_\Theta=-(-1)^{VW}\{W,V\}_\Theta$,

(2) $(-1)^{VZ}\{V,\{W,Z\}_\Theta\}_\Theta+(-1)^{WV}\{W,\{Z,V\}_\Theta\}_\Theta$
$+(-1)^{ZW}\{Z,\{V,W\}_\Theta\}_\Theta=0.$

Proof.

(1) We get the result by applying definition (30) to formula (31) rewritten in the form

$$(-1)^{VW}[\![W,d_\Theta V]\!]\backsim(-1)^V[\![V,d_\Theta W]\!]\,.$$

(2) The proof is deduced from the Jacobi identity for the Schouten–Nijenhuis bracket applied to multi-vector fields V, $d_\Theta W$ and $d_\Theta Z$:

$$(-1)^{VZ}[\![V, [\![d_\Theta W, d_\Theta Z]\!]]\!] + (-1)^{WV}[\![d_\Theta W, [\![d_\Theta Z, V]\!]]\!] \\ + (-1)^{ZW}[\![d_\Theta Z, [\![V, d_\Theta W]\!]]\!] = 0 .$$

It remains to rewrite the terms of this equality in the following way by making use of (31) and (32):

$$[\![V, [\![d_\Theta W, d_\Theta Z]\!]]\!] = [\![V, d_\Theta [\![W, d_\Theta Z]\!]]\!] \quad \Rightarrow \quad \{V, \{W, Z\}_\Theta\}_\Theta ,$$

$$[\![d_\Theta W, [\![d_\Theta Z, V]\!]]\!] \backsim (-1)^{W+Z}[\![W, d_\Theta [\![Z, d_\Theta V]\!]]\!] \\ \Rightarrow \quad (-1)^{W+Z}\{W, \{Z, V\}_\Theta\}_\Theta ,$$

$$[\![d_\Theta Z, [\![V, d_\Theta W]\!]]\!] \backsim (-1)^{Z}[\![Z, d_\Theta [\![V, d_\Theta W]\!]]\!] \\ \Rightarrow \quad (-1)^{Z}\{Z, \{V, W\}_\Theta\}_\Theta . \qquad \square$$

This proposition presents the Schouten–Nijenhuis bracket as the *differential* of the bracket $\{\cdot, \cdot\}_\Theta$ and, vice versa, the latter as the *integral* of the Schouten–Nijenhuis bracket.

Finally, we define the map

$$\gamma_\Theta : \mathscr{P}(M) \to \Pi_\Theta(M), \quad \gamma_\Theta([\omega]) = [\Gamma_\Theta(\omega)], \quad \text{for } \omega \in \Lambda^*(M) ,$$

where square brackets denote equivalence classes mod $B(M)$ and mod $\mathscr{B}_\Theta(M)$, respectively.

Proposition 31. *γ_Θ is a homomorphism of graded Lie algebras, i.e.,*

$$\gamma_\Theta\{[\omega], [\varrho]\} = \{\gamma_\Theta[\omega], \gamma_\Theta[\varrho]\}_\Theta .$$

Proof. The result follows from the definitions and from the properties of Γ_Θ listed in proposition 27:

$$\begin{aligned} \gamma_\Theta\{[\omega], [\varrho]\} &= \Gamma_\Theta([\![\omega, d\varrho]\!]_\Theta) (\mathrm{mod}\ \mathscr{B}_\Theta(M)) \\ &= [\![\Gamma_\Theta\omega, \Gamma_\Theta(d\varrho)]\!] (\mathrm{mod}\ \mathscr{B}_\Theta(M)) \\ &= [\![\Gamma_\Theta\omega, d_\Theta\Gamma_\Theta\varrho]\!] (\mathrm{mod}\ \mathscr{B}_\Theta(M)) \\ &= \{[\Gamma_\Theta\omega], [\Gamma_\Theta\varrho]\}_\Theta = \{\gamma_\Theta([\omega]), \gamma_\Theta([\varrho])\}_\Theta . \qquad \square \end{aligned}$$

Collecting now the above results we get:

Theorem 32. Diagram (29) consists of graded Lie algebras and their homomorphisms and is commutative. □

This work was performed in the framework of the programme "Geometria e fisica" supported by Ministero dell'Universitá e della Ricerca Scientifica e Tecnologica (local and national funds). This paper has been completed during the visit of Prof. A.M. Vinogradov at Dipartimento di Matematica Applicata "G. Sansone", Universitá di Firenze (supported by GNFM and GNSAGA of CNR).

Note added in proof

An alternative natural way to extend "hamiltonian" concepts to $\mathscr{P}(M)$ comes out by discarding the dogma that generalized hamiltonian fields should be generalized vector fields as well, say, elements of $N(M)$. Namely, let $|\cdot,\cdot|$ denote the unified bracket on $N(M)$ (see section 0 or ref. [11]). Define the hamiltonian operator ("field") Y_λ corresponding to $\lambda=[\omega]\in\mathscr{P}(M)$ to be

$$Y_\lambda = |\lambda, \Theta| = [\![\omega, \Theta]\!] \quad \mathrm{mod}\ \mathrm{Im}\, L \in \mathscr{P}(M) + N(M)\ .$$

The formula $Y_{\{\lambda,\mu\}} = |Y_\lambda, Y_\mu|$ shows that $\mathrm{Ham}\,\Theta = \{Y_\lambda \mid \lambda\in\mathscr{P}(M)\}$ is a graded Lie subalgebra of $\mathscr{P}(M)\oplus N(M) \subset \mathscr{N}(M)$. Next, we define analogs of canonical fields as operators $\Delta\in\mathscr{P}(M)\oplus N(M)$ such that $|\Delta, \Theta| = 0$. They form another graded Lie subalgebra of $\mathscr{P}(M)\oplus N(M)$ denoted by Can Θ. Then Ham Θ is an ideal of Can Θ. All these facts result directly from the Jacobi identity for the bracket $|\cdot,\cdot|$.

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